Seminar on Higher Structures: Derived Deformation Theory

The main goal of this seminar will be to understand Lurie's treatment of deformation theory as outlined in his ICM address [Lur10] and as worked out in detail in [Lur11]. In the first part, we will consider classical examples of deformation problems arising in algebra and geometry (see [Sze99] for a gentle introduction), before formalising them in the language of moduli problems as introduced by Schlessinger [Sch68]. We will observe that underlying each of our examples is a differential graded Lie algebra (dgla). This will lead us to understanding Lurie's central theorem on formal $\mathbb{E}_\infty$ moduli problems over a field k: that the ∞-category of moduli problems of this kind is equivalent to the ∞-category of dglas. The second part of the seminar will be devoted to understanding Lurie's approach to this statement; in particular, we will treat the tangent complex of a formal moduli problem and the concept of Koszul duality. The final talks will then survey Lurie's treatment of formal $\mathbb{E}_n$ moduli problems (the later sections of [Lur11]).

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Time and place

Monday, 10:15 – 11:45, Geom 434.

Preliminary program

Talk 1. (14 Oct, Severin) Introduction, overview, and organisation.

Talk 2. (21 Oct, Xinyang) Examples of deformations I: associative algebras.

Relate deformations and obstructions to Hochschild cohomology and Čech cohomology, following [Sze99, Ane, Fox93, Art]. Survey some of the examples from [CdSW99].

Talk 3. (28 Oct, Walker) Examples of deformations II: deformations of vector bundles and complex structures.

Survey deformations of vector bundles and Kodaira-Spencer theory [Huy05, Voi02, Ane, Sze99], unobstructedness of Calabi-Yau deformations.

Talk 4. (4 Nov) Formalisation of deformation problems: Schlessinger's deformation functors.

Explain the formalism introduced in [Sch68]. Reconsider the examples from Talk 2 (and Talk 3) from the point of view of deformation functors. Define the tangent to a moduli problem and show that it carries the structure of a vector space [Sch68, Ane, Sze99]. Explain first examples from [Lur10, Section 1].

Talk 5. (11 Nov) Differential graded Lie algebras and Maurer-Cartan theory.

Observe that in previous examples there are (dg) Lie algebra structures on the objects that control deformations (Gerstenhaber bracket on Hochschild homology as example [CdSW99]?). Survey the theory of dg Lie algebras and Maurer-Cartan theory, following [Man99] (first part, also [Sze99]). Explain how dglas give rise to moduli problems in the sense of Schlessinger [Man99] (second part, also [Sze99]).

Talk 6. (18 Nov) The idea of derived deformation theory and the motivation for Lurie's theorem.

Motivate allowing higher structure on both sides of Schlessinger's deformation functors: instead of functors from local Artin rings to sets, we should consider functors from local dg Artin rings to ∞-groupoids to describe *(derived) formal moduli problems.* Pass through groupoid-valued deformation problems, define (sketch) the tangent complex, and show that its π_0 carries a vector space structure [Gro, Ols, Ras], and even a dgla structure. Provide further motivation to use dg objects [Man04] (...) Explain Lurie's approach via $\mathbb{E}_\infty$-algebras in spectra?

Talk 7. (25 Nov) Deformation contexts and formal moduli problems.

Recall the most important notions from [Lur17, Sections 1.3, 1.4] (spectra as excisive functors), explaining how spectra generalise abelian groups and how $\mathbb{E}_\infty$-spectra generalise rings, prove [Lur17, Theorem 7.1.2.13]. Introduce Lurie's notion of a *deformation context* and of a *(derived) formal moduli problem* [Lur11, Section 1.1]. Emphasise the picture of [Lur11, Example 1.1.4] as a guiding principle (see also [Lur10, Section 3]).

Talk 8. (2 Dec) Tangent complex and deformation theories.

Present [Lur11, Sections 1.2 and 1.3].

Talk 9. (9 Dec) ∞-topoi and hypercoverings.

Introduce the notion of an ∞-topos via Giraud's axioms and as localisations of presheaf categories, following [Lur09, Chapter 6]. Define effective epimorphisms, slice ∞-topoi, and hypercoverings [Lur09, Sections 6.5.3 and 6.5.4].

Talk 10. (16 Dec) Deformation theories classify formal moduli problems.

Go through [Lur11, Sections 1.4 and 1.5] and present some of the arguments (especially from Section 1.5). Prove [Lur11, Theorem 1.3.12].

Talk 11. (6 Jan) Homology and cohomology of Lie algebras.

Present [Lur11, Sections 2.1 and 2.2].

Talk 12. (13 Jan) Koszul duality and the proof of the Lurie-Pridham Theorem [Lur11, Theorem 2.0.2].

Introduce the concept of Koszul duality, e.g. [ABCW, LV12], (dg) Lie algebras/L_∞-algebras and and cdgas/ as main example. Koszul duality for $\mathbb{E}_n$-operads [Lur17, Lur10, Lur11].

Talk 13. (20 Jan) Moduli problems for $\mathbb{E}_n$-algebras.

Survey [Lur11, Sections 3 and 4]

Talk 14. (27 Jan) Deformations of objects and categories.

[Lur11, Section 5].

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